

Jets, Lifts and Dynamics

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Abstract

We show that complete cotangent lifts of vector fields, their decomposition into vertical representative and holonomic part provide a geometrical framework underlying Eulerian equations of continuum mechanics. We discuss Euler equations for ideal incompressible fluid and Vlasov equations of plasma dynamics in connection with the lifts of divergence-free and Hamiltonian vector fields, respectively. As a further application, we obtain kinetic equations of particles moving with the flow of contact vector fields both from Lie-Poisson reductions and with the techniques of present framework.

Keywords: complete cotangent lift, vertical representative, diffeomorphism groups, kinetic equations of contact particles

1 Jets

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a smooth bundle with coordinates $(x^a; 1 \leq a \leq \dim(\mathcal{M}) = m)$ on the base manifold \mathcal{M} and $(x^a, u^\lambda; 1 \leq \lambda \leq \text{rank}(\pi) = k)$ on the total manifold \mathcal{E} . The vertical bundle associated with π is

$$V\pi = \ker T\pi = \{\xi \in T\mathcal{E} : T\pi(\xi) = 0\} \quad (1)$$

and this is a vector subbundle of the tangent bundle $T\mathcal{E}$. Here $T\pi$ denotes the tangent mapping of the projection π . Two sections $\phi, \psi \in \mathfrak{S}(\pi)$ of the bundle π at a point $\mathbf{x} \in \mathcal{M}$ are called equivalent if their tangent mappings are equal at that point, that is, $T_x\phi = T_x\psi$. Given a point \mathbf{x} , an equivalence class containing a section ϕ is denoted by $j_x^1\phi$. The first order jet manifold

$$J^1\pi = \{j_x^1\phi : \mathbf{x} \in \mathcal{M} \text{ and } \phi \in \mathfrak{S}(\pi)\} \quad (2)$$

associated with $(\mathcal{E}, \pi, \mathcal{M})$ is the set of equivalence classes at every point $\mathbf{x} \in \mathcal{M}$ with induced coordinates

$$(x^a, u^\lambda, u_a^\lambda) : J^1\pi \rightarrow \mathbb{R}^{m+k+mk} : j_x^1\phi \rightarrow \left(x^a, u^\lambda(\phi(\mathbf{x})), \left. \frac{\partial \phi^\lambda}{\partial x^a} \right|_x \right). \quad (3)$$

We have fibrations $\pi_0 : J^1\pi \rightarrow \mathcal{E} : j_x^1\phi \rightarrow \phi(\mathbf{x})$ and $\pi_1 : J^1\pi \rightarrow \mathcal{M} : j_x^1\phi \rightarrow \mathbf{x}$ of $J^1\pi$ on \mathcal{E} and \mathcal{M} , respectively [5],[15].

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Given a differentiable map $\rho : \mathcal{N} \rightarrow \mathcal{M}$ from a manifold \mathcal{N} to the base manifold \mathcal{M} , the pull-back bundle of π by ρ is the triple $(\rho^*\mathcal{E}, \rho^*\pi, \mathcal{N})$ where

$$\rho^*\mathcal{E} = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} = \{(\mathbf{n}, \mathbf{e}) \in \mathcal{N} \times \mathcal{E} : \pi(\mathbf{e}) = \rho(\mathbf{n})\} \quad (4)$$

is the Whitney product and, $\rho^*\pi = pr_1$ is the projection to the first factor [5]. Consider the pull back bundle

$$(\pi_0^*(T\mathcal{E}) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}, \pi_0^*\tau_{\mathcal{E}} = pr_1, J^1\pi)$$

of $(T\mathcal{E}, \tau_{\mathcal{E}}, \mathcal{E})$ by the projection $\pi_0 : J^1\pi \rightarrow \mathcal{E}$, where $\tau_{\mathcal{E}}$ is the tangent bundle projection. A section of $\pi_0^*\tau_{\mathcal{E}}$ is called a generalized vector field of order one [15],[16]. One may regard a section of $\pi_0^*\tau_{\mathcal{E}}$ as a map from $J^1\pi$ to $T\mathcal{E}$. We require that generalized vector fields are projectable [6].

In coordinates, a generalized vector field is

$$\xi(j_x^1\phi) = \xi^a(\mathbf{x}) \left. \frac{\partial}{\partial x^a} \right|_x + \xi^\lambda(j_x^1\phi) \left. \frac{\partial}{\partial u^\lambda} \right|_{\phi(x)} \quad (5)$$

and its first order prolongation $pr^1\xi$ is

$$pr^1\xi = \xi + \Phi_a^\lambda \frac{\partial}{\partial u_a^\lambda}, \quad \Phi_a^\lambda = D_{x^a}(\xi^\lambda - \xi^b u_b^\lambda) + \xi^b u_{ba}^\lambda \quad (6)$$

where D_{x^a} is the total derivative operator with respect to x^a and, $u_{ba}^\lambda(j_x\phi) = \partial^2\phi^\lambda/\partial x^a\partial x^b$ is an element of the second order jet bundle. Lie bracket of two first order generalized vector fields ξ and η is the unique first order generalized vector field

$$[\xi, \eta]_{pro} = (pr^1\xi(\eta^a) - pr^1\eta(\xi^a)) \frac{\partial}{\partial x^a} + (pr^1\xi(\eta^\lambda) - pr^1\eta(\xi^\lambda)) \frac{\partial}{\partial u^\lambda}. \quad (7)$$

If ξ and η are two vector fields on \mathcal{E} , then $[\ , \]_{pro}$ reduces to the Jacobi-Lie bracket of vector fields [13].

2 Lifts

Consider a vector field $X \in \mathfrak{X}(\mathcal{M})$ on \mathcal{M} , and let ϕ be a section of π . The holonomic lift of $X(\mathbf{x}) \in T_x\mathcal{M}$ by ϕ is

$$(j_x^1\phi, T\phi(X(\mathbf{x}))) \in \pi_0^*(T\mathcal{E}) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}. \quad (8)$$

In coordinates, if $X = X^a(\mathbf{x}) \partial/\partial x^a$, then

$$X^{hol} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial \phi^\lambda}{\partial x^a} \frac{\partial}{\partial u^\lambda} = X^a \frac{\partial}{\partial x^a} + X^a u_a^\lambda(j_x^1\phi) \frac{\partial}{\partial u^\lambda}. \quad (9)$$

Define the holonomic part of a projectable vector field $\xi \in \mathfrak{X}(\mathcal{E})$ as the holonomic lift of its push forward by π , that is

$$H\xi = (\pi_*\xi)^{hol}. \quad (10)$$

$H\xi$ is a generalized vector field of order one. Define a connection $(1;1)$ tensor

$$\Gamma_{\mathbf{J}} = dx^a \otimes \left(\frac{\partial}{\partial x^a} + u_a^\lambda \frac{\partial}{\partial u^\lambda} \right). \quad (11)$$

satisfying $H\xi = \Gamma_{\mathbf{J}}\xi$. Then, the vertical (or evolutionary) representative

$$V\xi = \xi - \Gamma_{\mathbf{J}}(\xi) = (\xi^\alpha - \xi^a u_a^\lambda) \frac{\partial}{\partial u^\lambda} \quad (12)$$

of ξ is vertical valued generalized vector field of order one [13],[15],[16].

Proposition 1 *Holonomic lift is a Lie algebra isomorphism from the space of projectable vector fields in $\mathfrak{X}(\mathcal{E})$ into $J^1\pi \times_{\mathcal{E}} T\mathcal{E}$.*

Proof. *We consider two projectable vector fields ξ and η on \mathcal{E} . A straight forward calculation gives*

$$[\Gamma_{\mathbf{J}}(\xi), \Gamma_{\mathbf{J}}(\eta)]_{pro} = [\xi^{hol}, \eta^{hol}]_{pro} = [\xi, \eta]^{hol} = \Gamma_{\mathbf{J}}[\xi, \eta] \quad (13)$$

where $[\ , \]_{pro}$ is the Lie bracket for generalized vector fields in Eq.(7). ■

On the other hand, the generalized bracket of vertical representatives satisfies

$$[V\xi, V\eta]_{pro} = V[\xi, \eta]_{pro} + \mathfrak{B}(\xi, \eta), \quad (14)$$

where \mathfrak{B} is a vertical-vector valued two-form

$$\mathfrak{B}(\xi, \eta) = [\eta^{hol}, V\xi]_{pro} - [\xi^{hol}, V\eta]_{pro}. \quad (15)$$

There is, however, a class of vector fields, defined again by lifts, for which the vertical representative becomes a Lie algebra isomorphism. Let $\varphi_t : \mathcal{M} \rightarrow \mathcal{M}$ be the flow of X on \mathcal{M} . Cotangent lift of φ_t is a one-parameter group of diffeomorphism φ_t^{c*} on $T^*\mathcal{M}$ satisfying

$$\pi_{\mathcal{M}} \circ \varphi_t^{c*} = \varphi_t \circ \pi_{\mathcal{M}} \quad (16)$$

where $\pi_{\mathcal{M}}$ is the natural projection of $T^*\mathcal{M}$ to \mathcal{M} . The cotangent lift of the inverse flow $T^*\varphi_{-t}$ satisfies the argument in Eq.(16). Infinitesimal generator $X^{c*} : T^*\mathcal{M} \rightarrow TT^*\mathcal{M}$ of the flow φ_t^{c*} is called complete cotangent lift of X . X^{c*} is a Hamiltonian vector field on the canonical symplectic manifold $(T^*\mathcal{M}, \Omega_{T^*\mathcal{M}} = -d\theta_{T^*\mathcal{M}})$ for the Hamiltonian function $P(X) = i_{X^{c*}}\theta_{T^*\mathcal{M}}$ [8]. The infinitesimal version

$$T\pi_{\mathcal{M}} \circ X^{c*} = X \circ \pi_{\mathcal{M}}.$$

of Eq.(16) gives the relation between X and X^{c*} with $T\pi_{\mathcal{M}}$ being the tangent mapping of $\pi_{\mathcal{M}}$. The complete cotangent lift mapping $^{c*} : \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(T^*\mathcal{M})$ taking X to X^{c*} is a Lie algebra isomorphism into [8],[17]

$$[X^{c*}, Y^{c*}] = [X, Y]^{c*}, \quad \forall X, Y \in \mathfrak{X}(\mathcal{M}). \quad (17)$$

In Darboux's coordinates (x^a, y_b) on $T^*\mathcal{M}$, the complete cotangent lift of $X = X^a(\mathbf{x}) \partial/\partial x^a$ on \mathcal{M} is

$$X^{c*} = X_{\mathcal{P}(X)} = X^a \frac{\partial}{\partial x^a} - y_b \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial y_a} \quad (18)$$

with the Hamiltonian function being $\mathcal{P}(X)(\mathbf{x}, \mathbf{y}) = y_b X^b(\mathbf{x})$. We decompose the complete cotangent lifts into vertical representative and holonomic part

$$VX^{c*} = -(y_b \frac{\partial X^b}{\partial x^a} + X^b \frac{\partial y_a}{\partial x^b}) \frac{\partial}{\partial y_a} \quad \text{and} \quad HX^{c*} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b}. \quad (19)$$

where the connection in Eq.(11) has the particular form

$$\Gamma = dx^a \otimes \left(\frac{\partial}{\partial x^a} + \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b} \right). \quad (20)$$

Proposition 2 *The mapping $V^{c*} : \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(T^*\mathcal{M}) : X \rightarrow VX^{c*}$ is a Lie algebra isomorphism into.*

Proof. *The vector valued two form \mathfrak{B} in Eq.(15) vanishes for the complete cotangent lifts, that is, $\mathfrak{B}(X^{c*}, Y^{c*}) = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$, therefore one has $V[X^{c*}, Y^{c*}] = [VX^{c*}, VY^{c*}]_{pro}$ and the result*

$$V[X, Y]^{c*} = [VX^{c*}, VY^{c*}]_{pro}. \quad (21)$$

follows from Eq.(17). ■

The last object we consider in this section is the vertical lift of one forms. Take the cotangent lift $T^*\pi_{\mathcal{M}} : T^*\mathcal{M} \rightarrow T^*T^*\mathcal{M}$ of the projection $\pi_{\mathcal{M}} : T^*\mathcal{M} \rightarrow \mathcal{M}$ and recall the isomorphism $\Omega_{T^*\mathcal{M}}^\sharp : T^*T^*\mathcal{M} \rightarrow TT^*\mathcal{M}$ associated with the symplectic two-form $\Omega_{T^*\mathcal{M}}$ on $T^*\mathcal{M}$. Define the Euler vector field

$$\mathcal{X}_E : T^*\mathcal{M} \rightarrow TT^*\mathcal{M} : \mathbf{z} \rightarrow \Omega_{T^*\mathcal{M}}^\sharp \circ T^*\pi_{\mathcal{M}}(\mathbf{z}) \quad (22)$$

which is vertical, that is, $image(\mathcal{X}_E) \subset ker(T\pi_{\mathcal{M}})$. Indeed,

$$\begin{aligned} \langle \mathbf{z}, T\pi_{\mathcal{M}} \circ \mathcal{X}_E(\mathbf{z}) \rangle &= \left\langle T^*\pi_{\mathcal{M}}(\mathbf{z}), \Omega_{T^*\mathcal{M}}^\sharp \circ T^*\pi_{\mathcal{M}}(\mathbf{z}) \right\rangle \\ &= \Omega_{T^*\mathcal{M}}(T^*\pi_{\mathcal{M}}(\mathbf{z}), T^*\pi_{\mathcal{M}}(\mathbf{z})) = 0, \end{aligned} \quad (23)$$

$\forall \mathbf{z} \in T^*\mathcal{M}$, where we used the skew-symmetry of $\Omega_{T^*\mathcal{M}}$. \mathcal{X}_E is the unique vector field satisfying the following equalities

$$i_{\mathcal{X}_E} \Omega_{T^*\mathcal{M}} = \theta_{T^*\mathcal{M}}, \quad \mathcal{L}_{\mathcal{X}_E} \Omega_{T^*\mathcal{M}} = -\Omega_{T^*\mathcal{M}}, \quad \mathcal{L}_{\mathcal{X}_E} \theta_{T^*\mathcal{M}} = -\theta_{T^*\mathcal{M}}, \quad (24)$$

where $i_{\mathcal{X}_E}$ and $\mathcal{L}_{\mathcal{X}_E}$ are the interior product and the Lie derivative operators [7]. Let $\alpha \in \Lambda^1(\mathcal{M})$ be a one-form on \mathcal{M} . The vertical lift

$$\alpha^v = \mathcal{X}_E \circ \alpha \circ \pi_{\mathcal{M}} : T^*\mathcal{M} \rightarrow TT^*\mathcal{M} \quad (25)$$

of the one-form α is a vertical vector field on $T^*\mathcal{M}$. The Jacobi-Lie bracket of a complete cotangent lift and a vertical lift is a vertical lift

$$[X^{c*}, \alpha^v] = (\mathcal{L}_X \alpha)^v \quad (26)$$

for $X \in \mathfrak{X}(\mathcal{M})$ and $\alpha \in \Lambda^1(\mathcal{M})$ [17]. In coordinates (x^a, y_b) of $T^*\mathcal{M}$, the Euler vector field is $\mathcal{X}_E = -y_a \partial / \partial y_a$ and the vertical lift of the one-form $\alpha = \alpha_a(\mathbf{x}) dx^a$ becomes $\alpha^v = -\alpha_a(\mathbf{x}) \partial / \partial y_a$.

3 Dynamics

Assume that a continuum initially rests in \mathcal{M} , and the group $Diff(\mathcal{M})$ of diffeomorphisms acts on left by evaluation on \mathcal{M}

$$Diff(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M} : (\varphi, \mathbf{x}) \rightarrow \varphi(\mathbf{x}) \quad (27)$$

to produce the motion of particles. The right action of $Diff(\mathcal{M})$ commutes with the particle motion and constitutes an infinite dimensional symmetry group of the kinematical description. This is the particle relabelling symmetry [2]. An element of the tangent space $T_\varphi Diff(\mathcal{M})$ at $\varphi \in Diff(\mathcal{M})$ is a map $V_\varphi : \mathcal{M} \rightarrow T\mathcal{M}$ called the material velocity field and satisfies $\tau_{\mathcal{M}} \circ V_\varphi = \varphi$. In particular, the tangent space $T_{id_{\mathcal{M}}} Diff(\mathcal{M})$ at the identity $id_{\mathcal{M}} \in Diff(\mathcal{M})$ is the space $\mathfrak{X}(\mathcal{M})$ of smooth vector fields on \mathcal{M} . The Lie algebra of $Diff(\mathcal{M})$ is $\mathfrak{X}(\mathcal{M})$ with minus the Jacobi-Lie bracket of vector fields [8].

The dual space $\mathfrak{X}^*(\mathcal{M}) \simeq \Lambda^1(\mathcal{M}) \otimes Den(\mathcal{M})$ of the Lie algebra is the space of one-form densities on \mathcal{M} . The pairing between $\alpha \otimes d\mu \in \mathfrak{X}^*(\mathcal{M})$ and $X \in \mathfrak{X}(\mathcal{M})$ is given by

$$\langle \alpha \otimes d\mu, X \rangle = \int_{\mathcal{M}} \langle \alpha(\mathbf{x}), X(\mathbf{x}) \rangle d\mu(\mathbf{x}). \quad (28)$$

The pairing inside the integral is the natural pairing of finite dimensional spaces $T_x \mathcal{M}$ and $T_x^* \mathcal{M}$. The coadjoint action is

$$\begin{aligned} ad_X^* &: \mathfrak{X}^*(\mathcal{M}) \rightarrow \mathfrak{X}^*(\mathcal{M}) \\ &: \alpha \otimes d\mu \rightarrow \mathcal{L}_X(\alpha \otimes d\mu) = (\mathcal{L}_X \alpha + (div_{d\mu} X) \alpha) \otimes d\mu \end{aligned} \quad (29)$$

$\forall X \in \mathfrak{X}(\mathcal{M})$ and hence the Lie-Poisson equations on $\mathfrak{X}^*(\mathcal{M})$ are

$$\dot{\alpha} = -\mathcal{L}_X \alpha - (div_{d\mu} X) \alpha, \quad (30)$$

where $div_{d\mu} X$ denotes the divergence of the vector field X with respect to the volume form $d\mu$.

In terms of vertical lifts, the dynamics in Eq.(30) is generated by the vector field $(\mathcal{L}_X \alpha + (div_{d\mu} X) \alpha)^v$. For the divergence free vector fields, if $\alpha = y_a dx^a$, then the Lie-Poisson equations are generated by

$$(\mathcal{L}_X(y_a dx^a))^v = V X^{c*}(x^a, y_a). \quad (31)$$

3.1 Ideal incompressible fluid

For an ideal incompressible fluid in a bounded compact region $\mathcal{Q} \subset \mathbb{R}^3$ the configuration space is the group $Diff_{vol}(\mathcal{Q})$ of volume preserving diffeomorphisms on \mathcal{Q} . The Lie algebra $\mathfrak{X}_{div}(\mathcal{Q})$ of $Diff_{vol}(\mathcal{Q})$ is the algebra of divergence free vector fields parallel to the boundary of \mathcal{Q} and, the dual space $\mathfrak{X}_{div}^*(\mathcal{Q})$ is the space

$$\mathfrak{X}_{div}^*(\mathcal{Q}) = \{[\Upsilon] \otimes d^3\mathbf{q} \in (\Lambda^1(\mathcal{Q})/d\mathcal{F}(\mathcal{Q})) \otimes Den(\mathcal{Q})\}, \quad (32)$$

of one-form modulo exact one-form densities on \mathcal{Q} . Here, $[\Upsilon] = \{\Upsilon + d\tilde{p} : \tilde{p} \in \mathcal{F}(\mathcal{Q})\}$ denotes the equivalence class containing Υ and the volume three form $d^3\mathbf{q}$ is the Euclidean volume on \mathbb{R}^3 [2],[11].

Let (x^a, Υ_b) be induced coordinates and $X = X^a \partial/\partial x^a$ be a divergence free vector field. The complete cotangent lift of X is

$$X^{c*} = X^a \frac{\partial}{\partial x^a} - \Upsilon_b (\partial X^b / \partial x^a) \frac{\partial}{\partial \Upsilon_a}$$

and its vertical representative becomes

$$VX^{c*} = \left(-\Upsilon_b \frac{\partial X^b}{\partial x^a} - X^a \frac{\partial \Upsilon_b}{\partial x^a} \right) \frac{\partial}{\partial \Upsilon_a}. \quad (33)$$

Equations of motion for the dynamics generated by VX^{c*} are

$$\frac{\partial [\Upsilon]}{\partial t} = -\mathcal{L}_X [\Upsilon]. \quad (34)$$

For a generic element $\Upsilon + d\tilde{p} \in [\Upsilon]$, Eq.(34) becomes Euler's equations for ideal fluid, that is $\partial \Upsilon / \partial t + \mathcal{L}_X \Upsilon = dp$. If the dual space $\mathfrak{X}_{div}^*(\mathcal{Q})$ is identified with exact two forms by $[\Upsilon] \rightarrow d\Upsilon = \omega \in \Lambda^2(\mathcal{Q})$, then Eq.(34) becomes the Euler's equation in vorticity form $\partial \omega / \partial t + \mathcal{L}_X \omega = 0$.

3.2 Collisionless plasma

We take \mathcal{M} to be cotangent bundle $T^*\mathcal{Q}$ of $\mathcal{Q} \subset \mathbb{R}^3$ in which the plasma particles move. The configuration space of collisionless nonrelativistic plasma is the group

$$Diff_{can}(T^*\mathcal{Q}) = \{\varphi \in T^*\mathcal{Q} : \varphi^* \Omega_{T^*\mathcal{Q}} = \Omega_{T^*\mathcal{Q}}\} \quad (35)$$

of all canonical diffeomorphisms where $\Omega_{T^*\mathcal{Q}}$ is the canonical symplectic two form on $T^*\mathcal{Q}$ [4],[9],[10]. We assume that, the Lie algebra of $Diff_{can}(T^*\mathcal{Q})$ is the space of globally Hamiltonian vector fields $\mathfrak{X}_{ham}(T^*\mathcal{Q})$ with minus the Jacobi-Lie bracket so that the equations

$$[X_h, X_f]_{JL} = -X_{\{h,f\}_{\Omega_{T^*\mathcal{Q}}}} \quad (36)$$

describe a Lie algebra isomorphism

$$h \rightarrow X_h : (\mathcal{F}(T^*\mathcal{Q}), \{, \}_{\Omega_{T^*\mathcal{Q}}}) \rightarrow (\mathfrak{X}_{ham}(T^*\mathcal{Q}), -[\cdot, \cdot]_{JL}), \quad (37)$$

between $\mathfrak{X}_{ham}(T^*\mathcal{Q})$ and the space of smooth functions $\mathcal{F}(T^*\mathcal{Q})$ modulo constants endowed with the (nondegenerate) canonical Poisson bracket $\{, \}_{\Omega_{T^*\mathcal{Q}}}$.

Proposition 3 *The dual space of the Lie algebra $\mathfrak{X}_{ham}(T^*\mathcal{Q})$ of Hamiltonian vector fields is*

$$\mathfrak{X}_{ham}^*(T^*\mathcal{Q}) = \{\Pi_{id} \otimes d\mu \in \Lambda^1(T^*\mathcal{Q}) \otimes Den(T^*\mathcal{Q}) : div_{\Omega_{T^*\mathcal{Q}}} \Pi_{id}^\# \neq 0\}. \quad (38)$$

With this definition of the dual space the L_2 -pairing of the Lie algebra and its dual becomes nondegenerate provided we take the volume form to be the symplectic one $d\mu = \Omega_{T^*\mathcal{Q}}^3$ in

$$\begin{aligned} \int_{T^*\mathcal{Q}} \langle X_h(\mathbf{z}), \Pi_{id}(\mathbf{z}) \rangle d\mu(\mathbf{z}) &= - \int_{T^*\mathcal{Q}} \langle dh, \Pi_{id}^\# \rangle d\mu = - \int_{T^*\mathcal{Q}} i_{\Pi_{id}^\#}(dh) d\mu \\ &= - \int_{T^*\mathcal{Q}} dh \wedge i_{\Pi_{id}^\#} d\mu = \int_{T^*\mathcal{Q}} h di_{\Pi_{id}^\#} d\mu \\ &= \int_{T^*\mathcal{Q}} h div_{\Omega_{T^*\mathcal{Q}}} \Pi_{id}^\# d\mu, \end{aligned} \quad (39)$$

where we use the musical isomorphism $\Omega_{T^*\mathcal{Q}}^\# : \Pi_{id} \rightarrow \Pi_{id}^\#$ induced from the symplectic two-form $\Omega_{T^*\mathcal{Q}}$ and apply integration by parts [8, internet supplement]. The dual of the Lie algebra isomorphism in Eq.(37) is

$$\Pi_{id}(\mathbf{z}) \rightarrow div_{\Omega_{T^*\mathcal{Q}}} \Pi_{id}^\#(\mathbf{z}) \quad (40)$$

and it is a momentum map. In Darboux's coordinates $\mathbf{z} = (q^i, p_i)$ on $T^*\mathcal{Q}$, we have $\Omega_{T^*\mathcal{Q}} = dq^i \wedge dp_i$ and we take $\Pi_{id} = \Pi_i(\mathbf{z}) dq^i + \Pi^i(\mathbf{z}) dp_i$. Then, the momentum map

$$f(\mathbf{z}) = div_{\Omega_{T^*\mathcal{Q}}} \Pi_{id}^\#(\mathbf{z}) = \frac{\partial \Pi^i(\mathbf{z})}{\partial q^i} - \frac{\partial \Pi_i(\mathbf{z})}{\partial p_i} \quad (41)$$

defines the plasma density function.

In the induced coordinates $(q^i, p_j; \Pi_i, \Pi^j)$ on $T^*T^*\mathcal{Q}$, consider the Hamiltonian function $h = (1/2m) \delta^{ij} p_i p_j + e\phi(\mathbf{q})$ which is the energy of a charged particle on \mathcal{Q} [9]. The corresponding Hamiltonian vector field is

$$X_h(\mathbf{z}) = \frac{1}{m} \delta^{ij} p_i \frac{\partial}{\partial q^j} - e \frac{\partial \phi}{\partial q^i} \frac{\partial}{\partial p_i}. \quad (42)$$

The complete cotangent lift of X_h and its decomposition into vertical representative and holonomic part are

$$\begin{aligned} X_h^{c*} &= X_h - \delta^{ij} \frac{1}{m} \Pi_i \frac{\partial}{\partial \Pi^j} + e \Pi^j \frac{\partial^2 \phi}{\partial q^j \partial q^i} \frac{\partial}{\partial \Pi_i}, \\ HX_h^{c*} &= X_h + X_h(\Pi_i) \frac{\partial}{\partial \Pi_i} + X_h(\Pi^i) \frac{\partial}{\partial \Pi^i}, \\ VX_h^{c*} &= \left(e \Pi^j \frac{\partial^2 \phi}{\partial q^j \partial q^i} - X_h(\Pi_i) \right) \frac{\partial}{\partial \Pi_i} - \left(\frac{1}{m} \Pi_j \delta^{ji} + X_h(\Pi^i) \right) \frac{\partial}{\partial \Pi^i}, \end{aligned} \quad (43)$$

where $X_h(\Pi_i)$ denotes the action of X_h on Π_i . Since, Hamiltonian vector fields are divergence free, the Lie-Poisson equations

$$\begin{aligned}\dot{\Pi}_i &= -X_h(\Pi_i) + e \frac{\partial^2 \phi}{\partial q^i \partial q^j} \Pi^j \\ \dot{\Pi}^i &= -X_h(\Pi^i) - \frac{1}{m} \delta^{ij} \Pi_j\end{aligned}\quad (44)$$

are generated solely by VX_h^{c*} . These are Vlasov equations in the momentum variables [4]. For the density formulation, we make back-substitution of the plasma density function $f(\mathbf{z}) = \text{div}_{\Omega_{T^*Q}} \Pi_{id}^\sharp$ into Eqs. (44) and obtain the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\delta^{ij} p_i}{m} \frac{\partial f}{\partial q^j} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} = 0. \quad (45)$$

3.3 Contact flows in 3D

Let \mathcal{M} be a three dimensional manifold with a contact one form $\sigma \in \Lambda^1(\mathcal{M})$ satisfying $d\sigma \wedge \sigma \neq 0$. A contact form determines a contact structure which, locally is the kernel of the contact form σ . A diffeomorphism on \mathcal{M} is called a contact diffeomorphism if it preserves the contact structure. We denote the group of contact diffeomorphisms by $\text{Diff}_{con}(\mathcal{M})$. A vector field on a contact manifold (\mathcal{M}, σ) is called a contact vector field if it generates a one-parameter group of contact diffeomorphisms [1],[12].

In Darboux's coordinates (x, y, z) on \mathcal{M} , we take the contact form to be $\sigma = xdy + dz$. For a real valued function $K = K(x, y, z)$ on \mathcal{M} , there corresponds a contact vector field

$$X_K = \left(\frac{\partial K}{\partial y} - x \frac{\partial K}{\partial z} \right) \frac{\partial}{\partial x} - \frac{\partial K}{\partial x} \frac{\partial}{\partial y} + \left(-K + x \frac{\partial K}{\partial x} \right) \frac{\partial}{\partial z}, \quad (46)$$

on \mathcal{M} satisfying the identities

$$i_{X_K} \sigma = -K \quad \text{and} \quad i_{X_K} d\sigma = dK - (i_{R_\sigma} dK) \sigma, \quad (47)$$

where $R_\sigma = \partial/\partial z$ is the Reeb vector field of σ . R_σ is the unique vector field satisfying $i_{R_\sigma} \sigma = 1$ and $i_{R_\sigma} d\sigma = 0$. The divergence $\text{div}_{d\mu} X_K$ of X_K with respect to the volume form $d\mu = d\sigma \wedge \sigma$ can be computed to be $\text{div}_{d\mu} X_K = -2R_\sigma K$.

Contact Poisson (or Lagrange) bracket of two smooth functions on \mathcal{M} is defined by

$$\{L, K\}_c = \frac{\partial L}{\partial x} \frac{\partial K}{\partial y} - \frac{\partial L}{\partial y} \frac{\partial K}{\partial x} + \frac{\partial K}{\partial z} \left(L - x \frac{\partial L}{\partial x} \right) - \frac{\partial L}{\partial z} \left(K - x \frac{\partial K}{\partial x} \right), \quad (48)$$

$\forall L, K \in \mathcal{F}(\mathcal{M})$. The identity $[X_K, X_L]_{JL} = -X_{\{K, L\}_c}$ establishes an isomorphism between Lie algebras $(\mathfrak{X}_{con}(\mathcal{M}), -[\cdot, \cdot]_{JL})$ and $(\mathcal{F}(\mathcal{M}), \{\cdot, \cdot\}_c)$. Following result gives a precise definition of the linear algebraic dual of $\mathfrak{X}_{con}(\mathcal{M})$.

Proposition 4 *The dual space of the algebra $\mathfrak{X}_{con}(\mathcal{M})$ of contact vector fields is*

$$\mathfrak{X}_{con}^*(\mathcal{M}) = \{\alpha \otimes d\mu \in \Lambda^1(\mathcal{M}) \otimes Den(\mathcal{M}) : d\alpha \wedge \sigma - 2\alpha \wedge d\sigma \neq 0\} \quad (49)$$

where σ is the contact form on \mathcal{M} and $d\mu = d\sigma \wedge \sigma$.

Proof. Proof. This follows from the requirement that the pairing between $\mathfrak{X}_{con}(\mathcal{M})$ and $\mathfrak{X}_{con}^*(\mathcal{M})$ be nondegenerate. We compute

$$\begin{aligned} \int_{\mathcal{M}} \langle \alpha, X_K \rangle d\mu &= \int_{\mathcal{M}} \alpha \wedge i_{X_K} d\sigma \wedge \sigma + \int_{\mathcal{M}} (i_{X_K} \sigma) \alpha \wedge d\sigma \\ &= \int_{\mathcal{M}} \alpha \wedge (dK - (i_{R_\sigma} dK) \sigma) \wedge \sigma - \int_{\mathcal{M}} K \alpha \wedge d\sigma \\ &= \int_{\mathcal{M}} K (d\alpha \wedge \sigma - 2\alpha \wedge d\sigma), \end{aligned} \quad (50)$$

where we use the identities in Eq.(47) at the second step. ■ ■

A geometric definition of density of contact particles can be achieved by considering the Lie algebra isomorphism $\mathcal{F}(\mathcal{M}) \rightarrow \mathfrak{X}_{con}(\mathcal{M}) : K \rightarrow X_K$ the dual of which is a momentum map

$$\mathfrak{X}_{con}^*(\mathcal{M}) \rightarrow Den(\mathcal{M}) : \alpha \rightarrow d\alpha \wedge \sigma - 2\alpha \wedge d\sigma. \quad (51)$$

and defines a real valued function L on \mathcal{M}

$$Ld\sigma \wedge \sigma = d\alpha \wedge \sigma - 2\alpha \wedge d\sigma. \quad (52)$$

In coordinates, let $\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz \in \mathfrak{X}_{con}^*(\mathcal{M})$ and recall $\sigma = xdy + dz$. Then,

$$L(x, y, z) = -\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x} - x \frac{\partial \alpha_z}{\partial x} + x \frac{\partial \alpha_x}{\partial z} - 2\alpha_z. \quad (53)$$

The dual space $\mathfrak{X}_{con}^*(\mathcal{N})$ admits the Lie-Poisson bracket

$$\{\mathfrak{H}, \mathfrak{K}\}(\alpha) = - \int_{\mathcal{M}} \left\langle \alpha, \left[\frac{\delta \mathfrak{H}}{\delta \alpha}, \frac{\delta \mathfrak{K}}{\delta \alpha} \right]_{JL} \right\rangle d\mu = - \int_{\mathcal{M}} \langle \alpha, [X_H, X_K]_{JL} \rangle d\mu, \quad (54)$$

where $\mathfrak{H}, \mathfrak{K} \in \mathcal{F}(\mathfrak{X}_{con}^*(\mathcal{M}))$ and $\delta \mathfrak{H}/\delta \alpha = X_H$, $\delta \mathfrak{K}/\delta \alpha = X_K \in \mathfrak{X}_{con}(\mathcal{M})$. The Hamiltonian operator $J_{LP}(\alpha)$ associated to the Lie-Poisson bracket in Eq.(54) is defined by

$$\{\mathfrak{H}, \mathfrak{K}\}(\alpha) = - \int_{\mathcal{M}} \langle X_H, J_{LP}(\alpha) X_K \rangle d\mu \quad (55)$$

and a direct computation gives

Proposition 5 *The Hamiltonian differential operator associated to the Lie-Poisson bracket in Eq.(54) is*

$$J_{LP}(\alpha) = - \begin{pmatrix} \alpha_x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \cdot \alpha_x & \alpha_y \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \cdot \alpha_x & \alpha_z \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \cdot \alpha_x \\ \alpha_x \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \cdot \alpha_y & \alpha_y \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \cdot \alpha_y & \alpha_z \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \cdot \alpha_y \\ \alpha_x \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot \alpha_z & \alpha_y \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \cdot \alpha_z & \alpha_z \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \cdot \alpha_z \end{pmatrix}, \quad (56)$$

where $\partial/\partial x \cdot \alpha_y = \alpha_y \partial/\partial x + \partial \alpha_y/\partial x$. Assuming $\delta \mathfrak{K}/\delta \alpha = X_K$, the Lie-Poisson equations on $\mathfrak{X}_{con}^*(\mathcal{M})$ are

$$\dot{\alpha} = J_{LP}(\alpha) X_K = -ad_{X_K}^* \alpha = -\mathcal{L}_{X_K} \alpha - (div_{d\mu} X_K) \alpha. \quad (57)$$

The Lie-Poisson bracket on the dual space $Den(\mathcal{M})$ of $\mathcal{F}(\mathcal{M})$, as defined by Eq.(51), is

$$\{\mathfrak{H}, \mathfrak{K}\}(L) = \int_{\mathcal{M}} L \left\{ \frac{\delta \mathfrak{H}}{\delta L}, \frac{\delta \mathfrak{K}}{\delta L} \right\}_c d\mu = \int_{\mathcal{M}} L \{H, K\}_c d\mu, \quad (58)$$

where $\delta \mathfrak{H}/\delta L = H$, $\delta \mathfrak{K}/\delta L = K \in \mathcal{F}(\mathcal{M})$ and $d\mu = d\sigma \wedge \sigma$.

Proposition 6 *The Hamiltonian operator $J_{LP}(L)$ for the Lie Poisson bracket in Eq.(54) is*

$$J_{LP}(L) = X_L + \left(4L + \frac{\partial L}{\partial z} \right) \frac{\partial}{\partial z}, \quad (59)$$

and the Lie-Poisson equation on $Den(\mathcal{M})$ becomes

$$\dot{L} = -\{L, K\}_c - 2div_{d\mu}(X_K)L. \quad (60)$$

Proof. *The verification of the Hamiltonian operator in Eq.(59) is a straightforward calculation which follows directly from the definition of the Lie-Poisson bracket in Eq.(54). To obtain the Lie-Poisson equation we compute the coadjoint action negative of which is the required equation. By definition*

$$\begin{aligned} \langle ad_K^* L, H \rangle &= \langle L, ad_K H \rangle = \langle L, \{K, H\}_c \rangle \\ &= - \int_{\mathcal{N}} L \{H, K\}_c d\mu = - \int_{\mathcal{N}} L \left(X_K(H) + \frac{\partial K}{\partial z} H \right) d\mu \\ &= \int_{\mathcal{N}} \left(X_K(L) + div_{d\mu}(X_K)L - \frac{\partial K}{\partial z} L \right) H d\mu \\ &= \int_{\mathcal{N}} \left(\{L, K\}_c - \frac{\partial K}{\partial z} L + div_{d\mu}(X_K)L - \frac{\partial K}{\partial z} L \right) H d\mu \\ &= \int_{\mathcal{N}} (\{L, K\}_c + 2div_{d\mu}(X_K)L) H d\mu, \end{aligned} \quad (61)$$

where we use integration by parts at the third step and the identities

$$\{H, K\}_c = X_K(H) + \frac{\partial K}{\partial z} H = -X_H(K) - \frac{\partial H}{\partial z} K \quad (62)$$

at the second and fourth steps. ■

The equation of motion $\dot{L} = -ad_K^* L$ is the kinetic equation of contact particles in density formulation.

Proposition 7 *The Hamiltonian differential operators $J_{LP}(\alpha)$ in Eq.(56) and $J_{LP}(L)$ in Eq.(59) are related by*

$$H J_{LP}(L) K = -X_H J_{LP}(\alpha) X_K \quad (mod \, div). \quad (63)$$

We now obtain dynamics of contact particles by the methods of previous sections. Let (\mathcal{M}, σ) be a contact manifold and consider the contact vector field X_K in Eq.(46). Its complete cotangent lift is

$$X_K^{c*} = X_K + \left(\Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} \right) \frac{\partial}{\partial \alpha_x} + (\Phi + \Psi) \left(\frac{\partial K}{\partial y} \frac{\partial}{\partial \alpha_y} + \frac{\partial K}{\partial z} \frac{\partial}{\partial \alpha_z} \right) \quad (64)$$

where we use the following abbreviations

$$\Upsilon = \alpha_x \left(1 + x \frac{\partial}{\partial x} \right), \quad \Psi = \alpha_y \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial y} - x \alpha_z \frac{\partial}{\partial x}, \quad \Phi = x \alpha_x \frac{\partial}{\partial z} + \alpha_z \quad (65)$$

and the induced coordinates $(x, y, z, \alpha_x, \alpha_y, \alpha_z)$ on $T^*\mathcal{N}$. X_K^{c*} is a canonically Hamiltonian vector field. The vertical representative VX_K^{c*} of X_K^{c*} is

$$\begin{aligned} VX_K^{c*} = & \left(\Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} - X_K(\alpha_x) \right) \frac{\partial}{\partial \alpha_x} + \left((\Phi + \Psi) \frac{\partial K}{\partial y} - X_K(\alpha_y) \right) \frac{\partial}{\partial \alpha_y} \\ & + \left((\Phi + \Psi) \frac{\partial K}{\partial z} - X_K(\alpha_z) \right) \frac{\partial}{\partial \alpha_z}, \end{aligned} \quad (66)$$

with $X_K(\alpha_x)$ denoting the action of X_K on α_x . To obtain the equations of motion for the momentum variables, one needs to add the divergence term, that is,

$$\dot{\alpha} = VX_K^{c*}(\alpha) - (\text{div}_{d\mu} X_K) \alpha. \quad (67)$$

It can be checked that Eq.(67) and Eq.(57) are equal. In coordinates, the system of equations in Eq.(67) takes the form

$$\begin{aligned} \dot{\alpha}_x &= \Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} - X_K(\alpha_x) + 2 \frac{\partial K}{\partial z} \alpha_x \\ \dot{\alpha}_y &= (\Phi + \Psi) \frac{\partial K}{\partial y} - X_K(\alpha_y) + 2 \frac{\partial K}{\partial z} \alpha_y \\ \dot{\alpha}_z &= (\Phi + \Psi) \frac{\partial K}{\partial z} - X_K(\alpha_z) + 2 \frac{\partial K}{\partial z} \alpha_z. \end{aligned} \quad (68)$$

Substituting L in Eq.(53) to the system of Eqs.(68) we obtain the evolution of the density of contact particles as given by Eq.(60).

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